Introduction to Twin Prime Clusters

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For a much
more comprehensive account
of Twin Prime Clusters, including an
ongoing list of QTPs which he has found
to date, about 62,000 in number as I write this,
go to D La Pierre Ballard's website at
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INTRODUCTION TO TWIN PRIME CLUSTERS

Primes

People have been fascinated by primes - integer numbers greater than 1 which are only divisible by themselves and 1 - since the days of Ancient Greece. Euclid proved that their number is infinite, and over the years mathematicians have discovered and proved that ever larger numbers are prime. From time to time the media report that another, bigger example has been found and proved to be prime, using large computers and sophisticated techniques. This has become an activity way beyond the expertise and equipment available to most of us, and we have to accept the word of professional mathematicians that the record has been broken.

However, there is one aspect of primes where an interested amateur can begin to gain some insight with the aid of a personal computer and little more than the mathematics learnt at school. This is the mystery of their distribution. It is too much to hope that a general explanation will be found. That has eluded the best brains tackling the problem over more than two millenia. Nevertheless, there are certain symmetrical groupings of primes whose analysis can elucidate, not a general rule or formula that will invariably produce examples of such groups, but at least some of the conditions which must be met for such examples to exist.

Twin primes

The simplest of these groups are called twin primes .They consist of two primes with a difference of 2, such as 11 and 13, or 29 and 31. The primes in each of these sets are separated by a composite number - one which is the product of two or more primes - which for want of a better name I call a pivotal composite , pivotal because it is the pivot of a symmetrical group of primes. The term pivotal composite, though descriptive, is inconveniently long, so in what follows it will be shortened to PIVCOM . It is well-known that all twin primes are of the form 6n-1 and 6n+1. All primes except 2 itself are odd numbers, and the two adjacent numbers can only be prime if the PIVCOM is divisible by three, since otherwise one of them would be a multiple of three. Therefore every pair of twin primes has a multiple of 6 as its PIVCOM. Note that although it is necessary for the PIVCOM to be a multiple of 6, it is not sufficient in all cases. For example, if n=4, 6n=24 and

6n+1=25, which is not a prime. (As a matter of interest, the lowest value of n for which neither of the adjacent numbers is prime is 20.)

Prime quadruplets

A larger symmetrical group is the prime quadruplet . These are made up of two sets of twin primes as close together as they can be. Here it turns out that the PIVCOM has to be a odd multiple of 15 and the differences of the primes in the group from the PIVCOM are -4, -2, +2,+4. The reason why this is so can be understood by using the concepts of modulus and residue . When one integer is divided by another (called a modulus), it either divides exactly to give a third integer, or else there is a residue or remainder. For example, 7 divides 35 exactly to give 5, but 38 divided by 7 gives 5 and a residue of 3. So we can say 38 to the modulus of 7 gives a residue of 3, or in short 38(mod 7) = 3. Two numbers which give the same residue to a modulus are said to be congruent with respect to that modulus.

Developing this idea a little further, if both a modulus and the residue of a certain number in respect of that modulus are divisible exactly by a third number, then the number itself must also be divisible by the third number. For example, 51(mod 9)=6; both 9 and 6 are divisible by 3; therefore 51 must itself be divisible by 3.

Now, we are looking for a PIVCOM which can sit between two sets of twin primes as closely as possible and equidistant from them. There are two possible sorts of PIVCOM for a symmetrical group of primes. If the PIVCOM is an odd number, then the distance from the centre of each prime must be a multiple of two, but if it is even, then the distances must be odd, because all primes except 2 itself are necessarily odd. It is fortunate that we are all accustomed to a number system based on ten, since it means that it is easy when thinking about primes to eliminate any number (except 2 and 5) which does not have as its last digit 1 or 3 or 7 or 9.

No three successive odd numbers n, n+2, n+4, where n is greater than 3, can all be prime, since one of them will always be divisible by 3. This means that sequences of successive odd numbers ending with the digits 3,7,9,1 or 9,1,3,7 or 7,9,1,3 necessarily contain at least one composite number.So we are left with 1,3,7,9 in that order as the last digits of a prime quadruplet. From this we can see that we need a PIVCOM ending in (and therefore divisible by) 5, with distances of -4,

-2, +2, +4. Consider now divisibility by 3. If the PIVCOM is congruent to 1 or 2(mod3) then either the +2 or the +4 number is divisible by 3 and not a prime. Therefore the PIVCOM must be congruent to 0(mod3), in other words, divisible by 3 as well as by 5. We have already seen that where the distances are even the PIVCOM must be odd, so we can also specify that it must be an odd multiple of 15. From this it follows that the nearest the pivcoms of two prime quadruplets can be to each other is 30. That is why a group of two prime number quadruplets is special and brings us to the main subject of this article.

PNQ30s

The abbreviation PNQ30 was coined to stand for “prime number quadruplets within 30 consecutive integers”, since PQ30 has meaning in other contexts. For convenience of listing, the first prime of the first quadruplet is used, and always ends with 1.

For a start, we know that the PIVCOM of a PNQ30 must be divisible by 30, since it is the even multiple of 15 which lies between two consecutive odd multiples of 15, namely the PIVCOMs of the two prime quadruplets. Secondly, since the PIVCOM of a PNQ30 differs from the PIVCOMs of its two quadruplets by -15 and +15 respectively, and each of these differ by -4, -2, +2 and +4 from the primes of which they are composed, it follows that the distances of the primes of a PNQ30 from its PIVCOM are -19, -17, -13, -11, +11, +13, +17, and +19. These distances imply that the PIVCOM of a PNQ30 is not divisible by 11, 13, 17 or 19, since divisibility by any one of these factors would mean that two of the numbers given by the distances, one each side of the PIVCOM, would also be composite. We have now accounted for the effect on the PIVCOM of all the primes up to 19, except one, namely 7. The PIVCOM is divisible by 2, 3 and 5, and not divisible by the others. Let us now look at how the PIVCOM of a PNQ30 stands in relation to divisibility by 7.

It was mentioned above that a number which has a residue in respect of a modulus which is divisible by a factor of that modulus is also divisible by that factor. There are seven possible residues in respect of modulus 7. these are 1, 2, 3, 4, 5, 6 and 0.

If PIVCOM(mod7) = 1, the potential prime with distance +13 from the PIVCOM would be congruent to 14(mod 7) and is therefore also divisible by seven and so not prime.

If PIVCOM(mod7) = 2, the potential prime with distance +19 would be congruent to 21(mod7).

If PIVCOM(mod7) = 3, the potential prime with distance +11 would be congruent to 14(mod7)
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If PIVCOM(mod7) = 4, the potential prime with distance -11 would be congruent to -7(mod7).

If PIVCOM(mod7) = 5, the potential prime with distance -19 would be congruent to -14(mod7).

If PIVCOM(mod7) = 6, the potential prime with distance -13 would be congruent to -7(mod7).

Therefore only if PIVCOM(mod7) = 0 are all the potential primes able to exist, so only in that case can a PNQ30 exist. Since this is equivalent to saying the PIVCOM must be divisible by 7, the requirement for a PIVCOM to be a multiple of 30 can be increased to a multiple of 210.

To summarise, a necessary requirement for a PNQ30 is that its PIVCOM must be a multiple of 210 but not of 11, 13, 17 or 19; and that the distances of the primes from the PIVCOM must be -19, -17,-13, -11, +11, +13, +17, and +19. As with the other PIVCOMs, it is not *sufficient* for this specification to be met to produce a PNQ30, but *unless it is met*, no PNQ30 can exist.

5TP39

Further consideration of the above led to the realisation that there was a possibilty that in rare cases, since the PIVCOM of a PNQ30 is necessarily divisible by 6, it might also itself be the PIVCOM of a twin prime. This would make five twin primes within 39 consecutive integers, or 5TP39 for short. They are also known as QTPs - Quintuplet twin primes.

This speculation was communicated to D La Pierre Ballard, the pioneer investigator of PNQ30s - and within a very few days he had discovered the smallest example of such a grouping. Very generously, he has chosen to name such clusters after me, but in my estimation they should be called Ballard Primes after this first discovery and his subsequent skill and diligence in discovering examples.